Pricing Stock and Bond Options in Incomplete Markets

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How should options be priced when securities markets are incomplete and price processes are unknown? Unlike modern options theory, the maximum-entropy framework offers process-free valuation in incomplete markets.[1] This article reviews the valuation models for stock and bond options derived in the maximum-entropy framework (Gulko [1999]). The models are valid in incomplete markets and do not restrict prices, interest rates, or volatilities to any specific trajectories. The models are both less restrictive and more accurate than alternative option valuation models.

The modern theory of option pricing was originally introduced in Black and Scholes [1973] and Merton [1973], and later refined in Ross [1976], Cox and Ross [1976], and Harrison and Kreps [1979]. It relies on two ideas. The first idea is the no-arbitrage principle that ensures the existence of universal risk-neutral probabilities and reduces asset valuation to the risk-free discounting of expected future values. The absence of arbitrage ensures only the existence, but not uniqueness or construction, of risk-neutral probabilities.

The second idea is that risk-neutral probabilities can be parameterized by a stochastic price process, which completes the market. In complete markets, there are unique risk-neutral probabilities and unique asset prices, while in incomplete markets this approach yields multiple risk-neutral probabilities and ambiguous prices. The real securities markets are not complete and market prices do not follow convenient paths. Modern valuation theory, therefore, uses unrealistic assumptions.

In contrast, the Entropy Pricing Theory (EPT) offers a process-free construction of unambiguous risk-neutral probabilities and asset prices in incomplete markets. The EPT formalizes the notion that efficient market prices maximize the collective uncertainty of investor expectations. When the collective uncertainty is measured by entropy, risk-neutral probabilities are the maximum-entropy consensus beliefs conditional on the information available at the valuation time.

The lognormal, gamma, and beta models are obtained in the EPT framework (Gulko [1999, 2002]) and reviewed in this article. The models are designed for pricing European-style options and named after their respective risk-neutral probability densities. The lognormal model is the entropic counterpart of the Black–Scholes model. The gamma model for pricing stock options reduces the Black–Scholes price biases and flattens the volatility smile. The beta model is a simple and practical tool for pricing fixed-income options.

THE MODELS

The lognormal, gamma, and beta models are designed to price European-style put and call options. Consider on the time interval

[0, T] an Arrow–Debreu market with a risky asset and a riskless pure discount bond that matures at time T. At the initial time 0, the price of the riskless bond P and the price of the risky asset S are known. At the terminal time T, the risk-free bond pays 1, while the risky asset price S_T is uncertain.

Consider a European-style option written on S_T. The option pays $h(S_T)$ at the expiration time T. A call option pays $h(S_T) = \max(0, S_T - K)$ and a put option pays $h(S_T) = \max(0, K - S_T)$, where K is the contractual strike price. It is required to evaluate the put and call options at time 0. By the no-arbitrage principle, the option value at time 0 is $P \cdot E_f[h(S_T)]$, where $E_f[\cdot]$ is the expectation with respect to a risk-neutral density $f(S_T)$, conditional on the information available at time 0. Thus, option valuation amounts to finding the risk-neutral density $f(S_T)$.

The state-of-the-art construction of risk-neutral densities relies on stochastic processes and market completeness. Stochastic processes parameterize risk-neutral densities. Complete markets guarantee a unique risk-neutral density $f(S_T)$. From a mathematical viewpoint, a complete market resembles a system of linear simultaneous equations that has a single solution. From an economic viewpoint, all possible risks in a complete market can be eliminated by hedging or insurance.

The actual securities markets are not complete and market prices do not follow convenient stochastic paths. Therefore, modern valuation theory uses unrealistic assumptions and often produces unrealistic prices.

Alternatively, a risk-neutral density $f(S_T)$ can be constructed in the EPT framework (Gulko [1999]). The EPT makes the efficient market hypothesis operational. Efficient market prices maximize the collective uncertainty about the future random price S_T. Let the entropy of consensus beliefs about S_T be the measure of collective uncertainty. Then, efficient market prices maximize the entropy, conditional on the information available at time 0. As a result, maximum-entropy consensus beliefs prevail in efficient markets.

In particular, if investors are risk neutral, then maximum-entropy risk-neutral consensus beliefs prevail in efficient markets. Therefore, a risk-neutral probability density $f(S_T)$ is determined by maximizing the entropy of S_T subject to the underlying price S, price volatility, risk neutrality, and other information available at the valuation time 0. There is usually a unique solution. The solution does not rely on complete markets or stochastic price processes.

Exhibit 1 details three maximum-entropy risk-neutral probability densities—lognormal, gamma, and beta. The densities are unique solutions to the maximum-entropy problems subject to the respective information sets. The information is explicitly described in the constraints. For example, the stock price S_T never falls below zero (due to the limited liability of equity investors), while a bond price is always confined between zero and some maximum value.

Exhibit 2 shows the lognormal, gamma, and beta valuation formulas. Although similar in appearance, the lognormal model is more general than the Black–Scholes model. The Black–Scholes model has a single source of risk, the stock price S. The lognormal model has three sources of risk—the stock price S, volatility σ_r, and price P of the riskless bond. A three-dimensional hedging policy is available in Gulko [1999].

THE GAMMA MODEL

The gamma model evaluates European-style stock options. It has four inputs—the stock price S, bond price P, strike price K, and volatility σ_γ. The model is subject to three sources of risk—S, P, and σ_γ—that are not restricted to any specific stochastic paths. The gamma option prices are homogeneous of degree one in (S, K, σ_γ), meaning that the computed option price is denominated in the same units as S, K, and σ_γ.

The risk-neutral gamma density $\gamma(S_T | u, v)$ is the defining feature of the gamma model. The gamma density is specified in Exhibit 1. Parameters u and v are determined in terms of the mean S/P and variance σ_γ^2 of the future stock price S_T.

Several gamma densities $\gamma(S_T | u, v)$ are plotted in Exhibit 3. The gamma density is more flexible than the lognormal density. When the parameters are $(u, v) = (6, 10)$, the gamma density has a bell shape like the lognormal density. When the parameters are $(u, v) = (2, 1)$, the gamma density is a declining exponential, unlike the lognormal density.

The empirical tests in Kim [1997] suggested that the gamma model is more accurate than the Black–Scholes model. We can establish that the gamma model reduces common Black–Scholes biases. First, the Black–Scholes model undervalues out-of-the-money put options. Second, the Black–Scholes prices of out-of-the-money puts are cheap relative to the Black–Scholes prices of in-the-money puts. Practitioners correct these *moneyness*

EXHIBIT 1
Risk-Neutral Probability Densities

	Lognormal Model (cf. Black-Scholes)	Gamma Model for Stock Options	Beta Model for Bond Options
Model Inputs	$S, P, K, \sigma_r^2 = Var[log(S_T/S)]$	$S, P, K, \sigma_\gamma^2 = Var(S_T)$	$S, P, K, Q, \sigma_\beta^2 = Var(S_T)$
Risk-Neutral Probability Density	$n(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ where $z = \frac{log(S_T/S) - \mu}{\sigma_r}$ $\mu = log\left(\frac{1}{P}\right) - \frac{\sigma_r^2}{2}$	$\gamma(S_T \mid u,v) = \frac{u^v}{\Gamma(v)} S_T^{v-1} e^{-uS_T}$ where $\Gamma(.)$ is the gamma function $u = \frac{(S/P)}{\sigma_\gamma^2}$ $v = \frac{(S/P)^2}{\sigma_\gamma^2}$	$\beta(s_T \mid u,v) = \frac{\Gamma(u+v)}{\Gamma(u)\Gamma(v)} (1-s_T)^{u-1} s_T^{v-1}$ where $s_T = S_T/Q$ $\Gamma(.)$ is the gamma function $u = (1-w)\left[\frac{w(1-w)}{(\sigma_\beta/Q)^2} - 1\right]$ $v = w\left[\frac{w(1-w)}{(\sigma_\beta/Q)^2} - 1\right]$ $w = \frac{1}{P}\left(\frac{S}{Q}\right)$
Cumulative Probability Function	$N(y) = \int_{-\infty}^{y} n(\xi) d\xi$ $N_1 = N(-d - 0.5\sigma_r)$, $\overline{N}_1 = 1 - N_1$ $N_2 = N(-d + 0.5\sigma_r)$, $\overline{N}_2 = 1 - N_2$ where $d \equiv \frac{1}{\sigma_r} \log\left(\frac{S}{PK}\right)$	$G(y \mid u,v) = \int_0^y \gamma(\xi \mid u,v) d\xi$ $G_1 = G(K \mid u, v+1)$, $\overline{G}_1 = 1 - G_1$ $G_2 = G(K \mid u, v)$, $\overline{G}_2 = 1 - G_2$	$B(y \mid u,v) = \int_0^y \beta(\xi \mid u,v) d\xi$ $B_1 = B(K/Q \mid u, v+1)$, $\overline{B}_1 = 1 - B_1$ $B_2 = B(K/Q \mid u, v)$, $\overline{B}_2 = 1 - B_2$

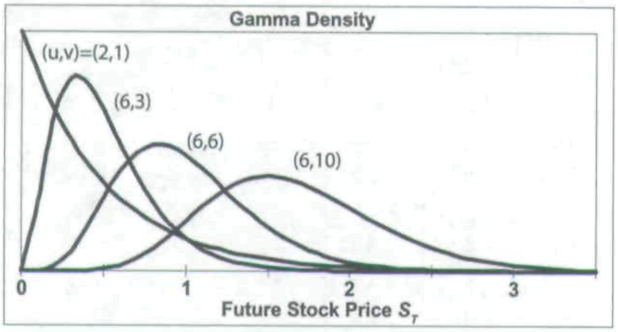
The risk-neutral probability densities—lognormal, gamma, and beta—of the random price S_T at the option expiration time T. The model inputs are the prices S, P, and K available at the valuation time 0, and respective volatility parameters σ_r (dimensionless), σ_γ (\$), and σ_β (\$). Q is the sum of all underlying bond payments scheduled after the option expiration date T.

EXHIBIT 2
EPT Option Pricing Models

	Lognormal Model (cf. Black-Scholes)	Gamma Model for Stock Options	Beta Model for Bond Options
Call Option Premium	$[S \quad -PK] \begin{bmatrix} \overline{N}_1 \\ \overline{N}_2 \end{bmatrix}$	$[S \quad -PK] \begin{bmatrix} \overline{G}_1 \\ \overline{G}_2 \end{bmatrix}$	$[S \quad -PK] \begin{bmatrix} \overline{B}_1 \\ \overline{B}_2 \end{bmatrix}$
Put Option Premium	$[-S \quad PK] \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$	$[-S \quad PK] \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$	$[-S \quad PK] \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$

The models are isomorphic to the Black–Scholes model. The probabilities N_1, N_2, G_1, G_2, B_1, B_2, and their complements are defined in Exhibit 1.

EXHIBIT 3
Gamma Probability Density



The risk-neutral gamma density $\gamma(S_T \mid u, v)$ of the future stock price S_T for different parameters u and v.

biases by adjusting the volatility parameter. The adjustment creates a *volatility smile* by making the volatility a declining function of the strike price K. Third, the volatility smile gets steeper as the volatility level rises.

Examine the gamma prices relative to the Black–Scholes prices listed in Exhibit 4. The prices are computed assuming that at the valuation time 0 the stock price is \$100 and the short-term risk-free rate is 5% per year. The dollar difference between the gamma and Black–Scholes prices is plotted in Exhibit 5 against the strike price K.

First, the gamma prices of out-of-the-money puts (i.e., when $K < 100$) uniformly exceed the corresponding Black–Scholes prices. Second, as the strike price K increases from left to right, the difference between the gamma and Black–Scholes prices gradually changes from positive to negative. This means that the gamma model has a smaller moneyness bias and flatter volatility smile than the Black–Scholes model. Third, the corrective action of the gamma model gets stronger as the volatility level rises. In sum, the gamma model is less biased than the Black–Scholes model.

THE BETA MODEL

The beta model evaluates European-style bond options. It has five inputs—the risky bond price S, risk-free bond price P, strike price K, volatility σ_β, and the scaling factor Q which is equal to the sum of the bond payments scheduled after the option expiration date T. The model has three sources of risk—S, P, and σ_β—that are not restricted to any specific stochastic paths. The beta option prices are homogeneous of degree one in (S, K, Q, σ_β), meaning that the computed option price is denominated in the same units as S, K, Q, and σ_β.

Suppose the underlying bond matures at time $m > T$. Let Q be the sum of nominal bond payments scheduled between the option expiration date T and the bond maturity date m. Then, the bond price S_T at time T ranges between 0 and Q. S_T may reach 0 if the bond issuer defaults and it may reach Q if the discount rates fall to zero.

Consider the normalized bond prices $s = S/Q$ at time 0 and $s_T = S_T/Q$ at time T. The normalized price s_T lies between 0 and 1. Therefore, the risk-neutral density of s_T is defined on the unit interval [0, 1]. The maximum-entropy risk-neutral density $f(s_T)$ is a standard beta density $\beta(s_T \mid u, v)$ with mean $(S/Q)/P$ and variance (σ_β/Q).[2] The beta density is defined in Exhibit 1.

Exhibit 6 displays several beta densities of the normalized bond price s_T when the option expires at time $T = 1$ year. The underlying bonds pay 10% annual interest and range in maturity from 3 to 30 years.

The bond maturity influences the shape of the risk-neutral beta density through the scaling factor Q. When the underlying is a long-term bond, the beta density in Exhibit 6 is skewed to the right like the lognormal density. When the underlying is a short-term bond, the beta density is skewed to the left, unlike the lognormal density. Therefore, option prices generated by the beta and Black–Scholes models should diverge most when the underlying is a short-term bond and diverge least when the underlying is a long-term bond.

The empirical tests in Gulko [2007] concluded that the beta model is more accurate than Black–Scholes [1973], Black [1976], Brennan [1979], Jamshidian [1989], and several versions of the Heath–Jarrow–Morton [1992] model. The tests utilized over 32,000 prices of Eurodollar futures options and demonstrated that the beta model has an average absolute pricing error under one basis point (the smallest price change and the bid–ask spread), while the other models have average absolute pricing errors over one basis point.

We can establish that the beta model reduces common price biases generated by the Black–Scholes model and other option pricing models. First, the Black–Scholes model undervalues out-of-the-money put options on bonds and causes a volatility smile discussed in the last section of this article. Second, the price bias is stronger for long-dated options than for short-dated options. Third, the price bias is stronger when the underlying is a short-term bond and weaker when the underlying is a long-term bond.[2]

EXHIBIT 4

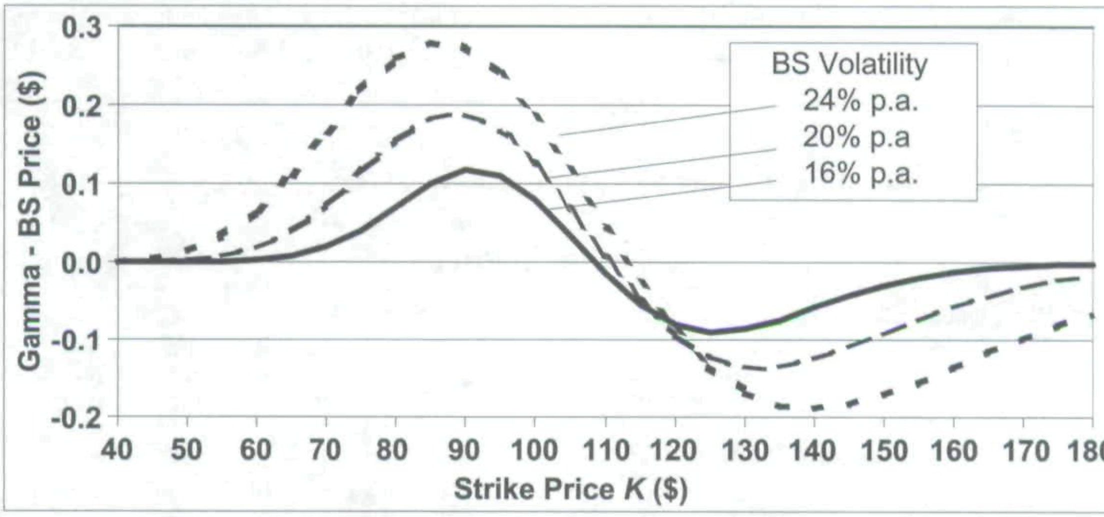
Gamma and Black–Scholes Prices of Stock Options

Option Term	BS Vol	Option Model	80	85	90	95	100	105	110	115	120
						Strike Price K					
						Call Option Prices					
0.25 yr	16%	*Gamma*	21.00	16.10	11.37	7.14	3.84	1.72	0.63	0.18	0.04
		BS	21.00	16.09	11.35	7.12	3.83	1.73	0.65	0.20	0.05
	20%	*Gamma*	21.03	16.23	11.71	7.75	4.63	2.46	1.16	0.48	0.17
		BS	21.02	16.20	11.67	7.71	4.61	2.48	1.19	0.51	0.20
	24%	*Gamma*	21.12	16.46	12.16	8.42	5.43	3.23	1.77	0.89	0.42
		BS	21.09	16.41	12.09	8.37	5.40	3.25	1.82	0.95	0.46
0.5 yr	16%	*Gamma*	22.04	17.34	12.93	9.02	5.83	3.46	1.87	0.92	0.41
		BS	22.02	17.31	12.88	8.97	5.80	3.46	1.91	0.97	0.46
	20%	*Gamma*	22.23	17.73	13.59	9.95	6.93	4.58	2.87	1.70	0.95
		BS	22.17	17.65	13.50	9.87	6.89	4.58	2.91	1.76	1.02
	24%	*Gamma*	22.55	18.27	14.37	10.94	8.05	5.72	3.92	2.59	1.65
		BS	22.45	18.15	14.24	10.94	7.99	5.71	3.96	2.67	1.75
1 yr	16%	*Gamma*	24.22	19.86	15.81	12.17	9.04	6.47	4.45	2.95	1.87
		BS	24.15	19.76	15.69	12.06	8.96	6.43	4.47	3.00	1.96
	20%	*Gamma*	24.74	20.65	16.89	13.52	10.58	8.09	6.05	4.42	3.15
		BS	24.59	20.47	16.70	13.35	10.45	8.02	6.04	4.47	3.25
	24%	*Gamma*	24.49	21.64	18.11	14.94	12.14	9.72	7.67	5.96	4.57
		BS	24.23	21.36	17.84	14.70	11.96	9.61	7.63	5.99	4.66
						Put Option Prices					
0.25 yr	16%	*Gamma*	0.01	0.05	0.26	0.96	2.60	5.41	9.26	13.76	18.55
		BS	0.00	0.04	0.23	0.94	2.59	5.43	9.28	13.77	18.56
	20%	*Gamma*	0.04	0.17	0.59	1.57	3.39	6.16	9.79	14.05	18.68
		BS	0.03	0.15	0.55	1.53	3.37	6.17	9.82	14.09	18.71
	24%	*Gamma*	0.13	0.40	1.04	2.24	4.18	6.93	10.41	14.47	18.92
		BS	0.10	0.35	0.97	2.19	4.16	6.94	10.45	14.52	18.97
0.5 yr	16%	*Gamma*	0.07	0.25	0.71	1.68	3.36	5.86	9.16	13.08	17.45
		BS	0.05	0.21	0.65	1.62	3.33	5.87	9.19	13.13	17.49
	20%	*Gamma*	0.25	0.63	1.37	2.61	4.47	6.99	10.15	13.86	17.99
		BS	0.20	0.55	1.28	2.53	4.42	6.99	10.19	13.92	18.06
	24%	*Gamma*	0.58	1.17	2.15	3.60	5.58	8.13	11.20	14.75	18.69
		BS	0.48	1.05	2.02	3.49	5.52	8.11	11.24	14.83	18.79
1 yr	16%	Gamma	0.32	0.72	1.42	2.54	4.16	6.35	9.09	12.34	16.02
		BS	0.25	0.62	1.30	2.43	4.08	6.31	9.10	12.39	16.10
	20%	Gamma	0.84	1.51	2.50	3.88	5.70	7.97	10.68	13.81	17.30
		BS	0.69	1.32	2.31	3.71	5.57	7.90	10.68	13.86	17.40
	24%	Gamma	1.59	2.50	3.73	5.31	7.27	9.60	12.31	15.36	18.72
		BS	1.33	2.22	3.45	5.07	7.08	9.49	12.26	15.38	18.81

At the valuation time 0, the stock price is \$100 and the short-term interest rate is 5% per year. (BS Vol) is the annual Black–Scholes volatility of stock return. The gamma volatility σ_γ of the future stock price S_T corresponding to the cumulative Black–Scholes volatility (BS Vol) $\times \sqrt{T}$ is determined as follows. If a random variable $\log(X)$ is normally distributed with mean a and standard deviation b, then X is lognormally distributed with mean $\exp(a + 0.5b^2)$ and variance $\exp(2a + b^2)[\exp(b^2) - 1]$.

EXHIBIT 5

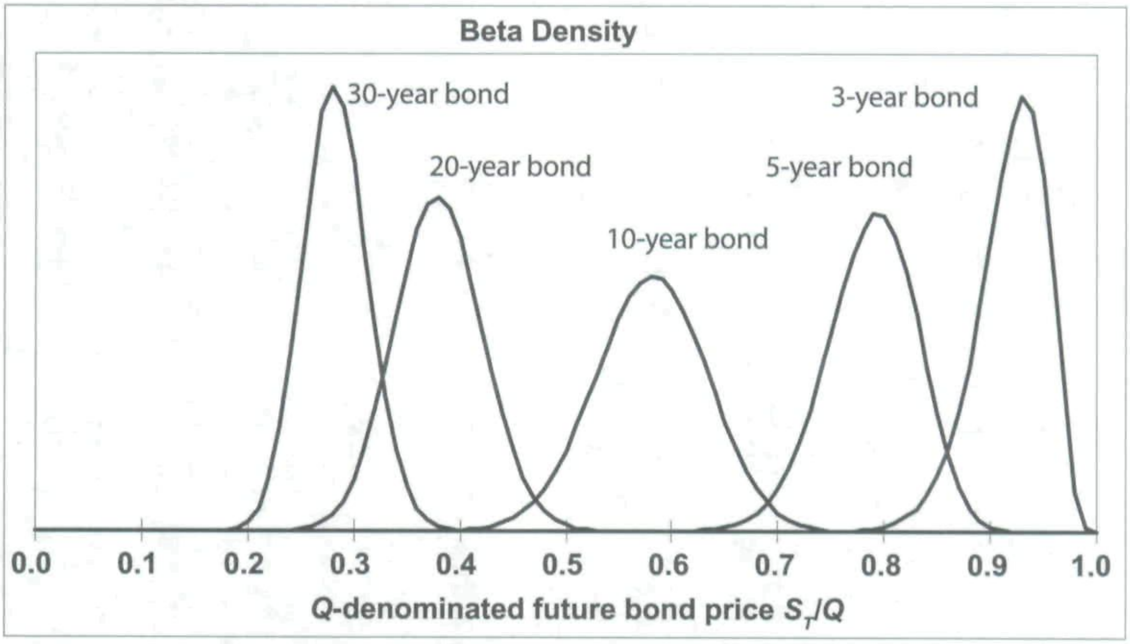
Difference between Gamma and Black–Scholes Prices of One-Year Stock Options



At the valuation time, the underlying stock price is \$100 and the short-term risk-free rate is 5% per year. The options expire in one year. The Black–Scholes volatilities are 16%, 20%, and 24% per year. The gamma prices exceed the Black–Scholes prices for strike prices $K < 100$. Greater volatilities amplify the price difference. The price differences are the same for puts and calls by the put–call parity.

EXHIBIT 6

Beta Probability Density



The risk-neutral beta densities $\beta(S_T/Q \mid u, v)$ of the random bond price S_T when $T = 1$ year. The bonds mature in 3, 5, 10, 20, and 30 years; pay \$100 at maturity; and pay 10% interest per year. Q is the sum of all nominal bond payments from time T until the bond maturity date m. For example, when $T = 1$ year and $m = 3$ years, $Q = 100 + (m - T) \times 10 = 220$. Since S_T ranges between 0 and Q, S_T/Q ranges between 0 and 1.

Examine the beta and Black–Scholes prices listed in Exhibits 7 and 8. The underlying securities are 3-, 10-, and 30-year bonds paying 10% annual interest and priced at \$100 at the valuation time. The short-term risk-free rate is 10% per year. The difference between the beta and Black–Scholes prices of one-year options is plotted in Exhibit 9 against the strike price K. The price difference represents the beta correction to the Black–Scholes prices.

First, the beta prices of out-of-the-money puts (i.e., when $K < 100$) exceed the corresponding Black–Scholes

EXHIBIT 7

Beta and Black–Scholes Prices of Call Options on 3-, 10-, and 30-Year Bonds

Call Option Prices

Option Term	BS Vol	Option Model	Bond Term	*Strike Price K* 80	85	90	95	100	105	110	115	120
0.25 yr	8%	*Beta*	3 yr	21.98	17.10	12.23	7.42	3.15	0.64	0.03	0.00	0.00
			10 yr	21.98	17.10	12.22	7.40	3.13	0.67	0.05	0.00	0.00
			30 yr	21.98	17.10	12.22	7.39	3.12	0.68	0.06	0.00	0.00
		BS	any	21.98	17.10	12.22	7.39	3.11	0.69	0.07	0.00	0.00
	10%	*Beta*	3 yr	21.98	17.10	12.25	7.55	3.51	0.98	0.11	0.00	0.00
			10yr	21.98	17.10	12.23	7.50	3.47	1.02	0.16	0.01	0.00
			30yr	21.98	17.10	12.23	7.49	3.46	1.03	0.17	0.02	0.00
		BS	any	21.98	17.10	12.23	7.48	3.45	1.04	0.19	0.02	0.00
	12%	*Beta*	3 yr	21.98	17.10	12.30	7.74	3.88	1.33	0.25	0.02	0.00
			10 yr	21.98	17.10	12.27	7.67	3.83	1.38	0.33	0.05	0.00
			30 yr	21.98	17.10	12.26	7.65	3.82	1.39	0.35	0.06	0.01
		BS	any	21.98	17.10	12.25	7.62	3.80	1.41	0.37	0.07	0.01
0.5 yr	8%	*Beta*	3 yr	19.15	14.41	9.79	5.56	2.32	0.57	0.06	0.00	0.00
			10 yr	19.15	14.40	9.74	5.49	2.32	0.66	0.11	0.01	0.00
			30 yr	19.15	14.40	9.72	5.47	2.32	0.68	0.13	0.01	0.00
		BS	any	19.15	14.39	9.69	5.38	2.21	0.62	0.11	0.01	0.00
	10%	*Beta*	3 yr	19.17	14.50	10.02	6.02	2.89	0.97	0.18	0.01	0.00
			10 yr	19.15	14.44	9.92	5.91	2.89	1.09	0.30	0.06	0.01
			30 yr	19.15	14.43	9.89	5.88	2.89	1.12	0.33	0.07	0.01
		BS	any	19.15	14.41	9.82	5.75	2.75	1.03	0.30	0.07	0.01
	12%	*Beta*	3 yr	19.24	14.67	10.35	6.52	3.46	1.41	0.37	0.04	0.00
			10 yr	19.18	14.55	10.18	6.38	3.46	1.57	0.57	0.16	0.04
			30 yr	19.17	14.52	10.14	6.34	3.45	1.60	0.62	0.20	0.05
		BS	any	19.15	14.47	10.02	6.16	3.28	1.49	0.57	0.19	0.05
1 yr	8%	*Beta*	3 yr	18.44	14.09	9.98	6.30	3.32	1.30	0.29	0.02	0.00
			10 yr	18.36	13.95	9.78	6.13	3.32	1.50	0.54	0.15	0.03
			30 yr	18.35	13.92	9.74	6.09	3.32	1.53	0.59	0.18	0.05
		BS	any	18.34	13.86	9.57	5.81	3.01	1.30	0.46	0.14	0.03
	10%	*Beta*	3yr	18.67	14.49	10.57	7.06	4.12	1.94	0.61	0.07	0.00
			10yr	18.47	14.20	10.26	6.84	4.13	2.22	1.04	0.41	0.14
			30yr	18.43	14.14	10.18	6.78	4.12	2.25	1.10	0.48	0.18
		BS	any	18.37	13.98	9.90	6.40	3.73	1.95	0.91	0.38	0.14
	12%	*Beta*	3 yr	19.05	15.02	10.95	7.84	4.92	2.61	1.01	0.18	0.00
			10 yr	18.67	14.57	10.83	7.58	4.95	2.97	1.62	0.80	0.34
			30 yr	18.60	14.48	10.73	7.51	4.93	3.01	1.71	0.90	0.43
		BS	any	18.45	14.20	10.32	7.03	4.45	2.62	1.44	0.73	0.35

The bonds pay \$100 at maturity and 10% interest per year. At the valuation time 0, all bonds are priced at \$100 and the short-term risk-free rate is 10% per year. The option values are expressed as a percent of par. (BS Vol) is the annual Black–Scholes volatility of bond return. The cumulative Black–Scholes volatility over the term to option expiration T is (BS Vol) × $\sqrt{T}$. The volatility σ_β of the future bond price S_T consistent with the cumulative Black–Scholes volatility is determined as follows. If a random variable log(X) is normally distributed with mean a and standard deviation b, then X is lognormally distributed with mean $\exp(a + 0.5b^2)$ and variance $\exp(2a + b^2)[\exp(b^2)-1]$.

EXHIBIT 8

Beta and Black–Scholes Prices of Put Options on 3-, 10-, and 30-Year Bonds

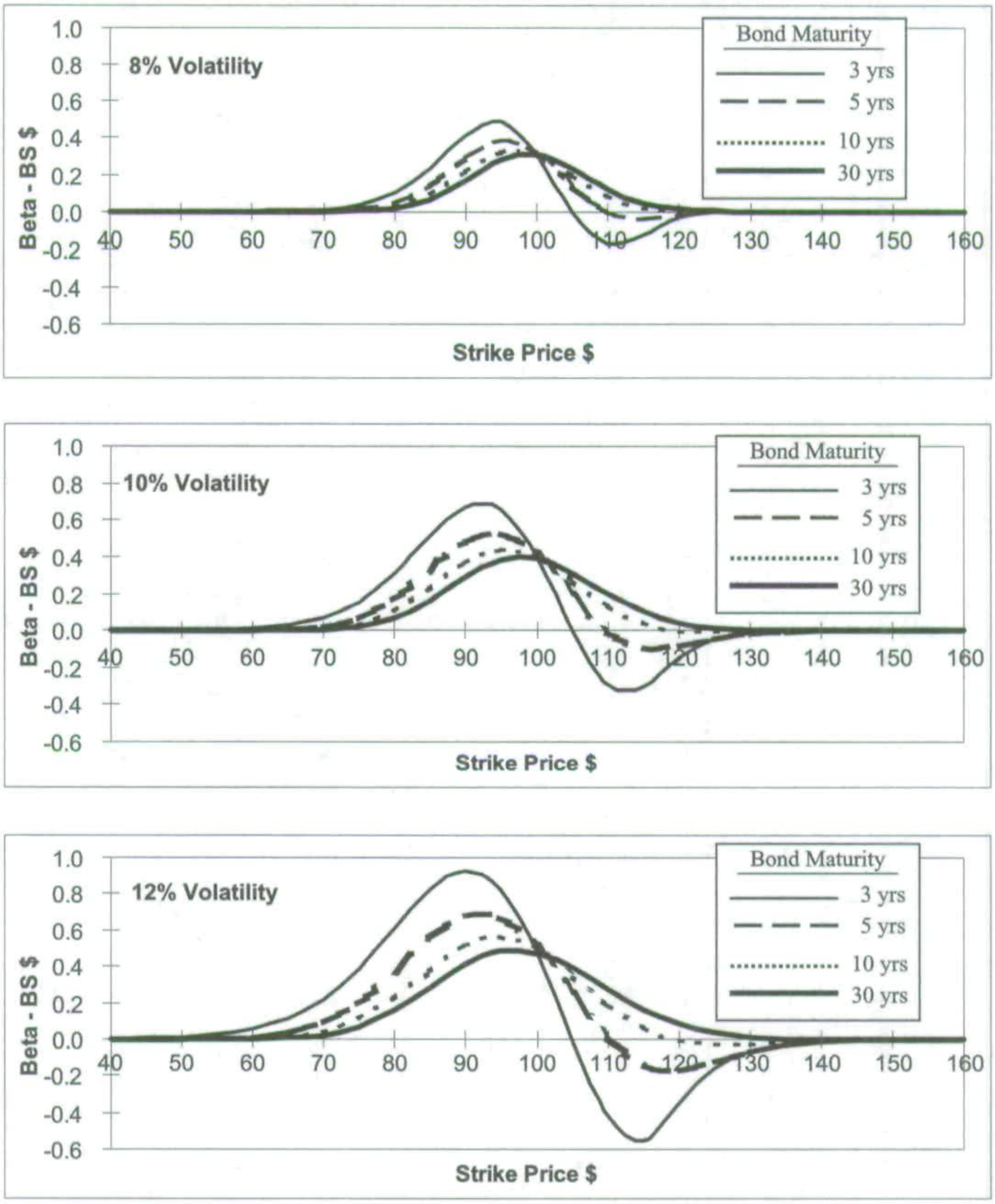
Put Option Prices

Option Term	*BS* Vol	Option Model	Bond Term	*Strike Price K* 80	85	90	95	100	105	110	115	120
0.25 yr	8%	*Beta*	3 yr	0.00	0.00	0.00	0.07	0.69	3.05	7.32	12.16	17.04
			10 yr	0.00	0.00	0.00	0.05	0.66	3.08	7.34	12.16	17.04
			30 yr	0.00	0.00	0.00	0.05	0.65	3.09	7.34	12.16	17.04
		BS	any	0.00	0.00	0.00	0.04	0.64	3.10	7.35	12.16	17.04
	10%	*Beta*	3 yr	0.00	0.00	0.02	0.20	1.04	3.39	7.40	12.18	17.04
			10 yr	0.00	0.00	0.01	0.16	1.00	3.43	7.43	12.21	17.04
			30 yr	0.00	0.00	0.01	0.15	0.99	3.44	7.46	12.22	17.04
		BS	any	0.00	0.00	0.01	0.13	0.98	3.45	7.47	12.23	17.05
	12%	*Beta*	3 yr	0.00	0.01	0.08	0.39	1.41	3.74	7.53	12.18	17.04
			10 yr	0.00	0.00	0.05	0.32	1.36	3.79	7.61	12.21	17.04
			30 yr	0.00	0.00	0.04	0.30	1.35	3.80	7.63	12.22	17.04
		BS	any	0.00	0.00	0.03	0.28	1.33	3.81	7.65	12.23	17.05
0.5 yr	8%	*Beta*	3 yr	0.00	0.03	0.16	0.69	2.20	5.21	9.45	14.15	18.90
			10 yr	0.00	0.01	0.10	0.61	2.20	5.29	9.50	14.16	18.90
			30 yr	0.00	0.01	0.09	0.59	2.20	5.31	9.52	14.16	18.90
		BS	any	0.00	0.00	0.06	0.50	2.09	5.25	9.50	14.16	18.90
	10%	*Beta*	3 yr	0.03	0.11	0.39	1.14	2.77	5.61	9.57	14.16	18.90
			10 yr	0.01	0.05	0.28	1.03	2.77	5.73	9.69	14.21	18.91
			30 yr	0.00	0.04	0.26	1.01	2.76	5.75	9.72	14.22	18.92
		BS	any	0.00	0.02	0.18	0.87	2.62	5.67	9.69	14.22	18.92
	12%	*Beta*	3 yr	0.09	0.28	0.72	1.64	3.34	6.05	9.76	14.19	18.90
			10 yr	0.03	0.16	0.55	1.50	3.34	6.20	9.96	14.31	18.94
			30yr	0.02	0.13	0.51	1.46	3.33	6.23	10.01	14.34	18.95
		BS	any	0.01	0.08	0.38	1.29	3.16	6.12	9.96	14.33	18.96
1 yr	8%	*Beta*	3 yr	0.10	0.28	0.70	1.54	3.09	5.59	9.10	13.35	17.86
			10 yr	0.03	0.14	0.49	1.37	3.09	5.79	9.35	13.49	17.89
			30 yr	0.02	0.11	0.45	1.33	3.08	5.82	9.40	13.52	17.91
		BS	any	0.00	0.05	0.28	1.05	2.77	5.59	9.47	13.47	17.89
	10%	*Beta*	3 yr	0.34	0.68	1.28	2.30	3.89	6.23	9.42	13.41	17.86
			10 yr	0.13	0.39	0.97	2.08	3.90	6.51	9.85	13.75	18.00
			30 yr	0.10	0.33	0.90	2.02	3.88	6.54	9.92	13.82	18.04
		BS	any	0.03	0.17	0.61	1.64	3.50	6.24	9.72	13.72	18.00
	12%	*Beta*	3 yr	0.72	1.21	1.96	3.08	4.69	6.90	9.82	13.52	17.86
			10 yr	0.34	0.76	1.54	2.82	4.71	7.26	10.44	14.13	18.21
			30 yr	0.27	0.67	1.44	2.75	4.69	7.30	10.52	14.23	18.29
		BS	any	0.11	0.39	1.04	2.27	4.22	6.91	10.25	14.07	18.21

The bonds pay \$100 at maturity and 10% interest per year. At the valuation time 0, all bonds are priced at \$100 and the short-term risk-free rate is 10% per year. The option values are expressed as a percent of par. (BS Vol) is the annual Black–Scholes volatility of bond return. The cumulative Black–Scholes volatility over the term to option expiration T is (BS Vol) × $\sqrt{T}$. The volatility σ_β of the future bond price S_T consistent with the cumulative Black–Scholes volatility is determined as follows. If a random variable log(X) is normally distributed with mean a and standard deviation b, then X is lognormally distributed with mean $\exp(a + 0.5b^2)$ and variance $\exp(2a + b^2)[\exp(b^2)-1]$.

EXHIBIT 9

Difference between Beta and Black–Scholes Prices of One-Year Options on 3-, 5-, 10-, and 30-Year Bonds



The bonds pay \$100 at maturity and 10% interest per year. At the valuation time 0, all bonds are priced at \$100 and the short-term risk-free rate is 10% per year. The options expire in one year. The annualized Black–Scholes volatility is 8% in the top panel, 10% in the middle panel, and 12% in the bottom panel. The beta prices exceed the Black–Scholes prices for strike prices $K < 100$. Greater volatilities and shorter bond maturities amplify the price difference. The price differences are the same for puts and calls by the put–call parity.

prices. As the strike price K increases from left to right, the price difference gradually changes from positive to negative. Therefore, the beta model has a smaller moneyness bias and flatter volatility smile than the Black–Scholes model. Second, the price differences are greater for long-dated options (characterized by larger σ_β) than for short-dated options. Third, the price differences are greater for the options on short-term bonds than for the options on long-term bonds. In sum, the beta model reduces the common model biases.

Finally, the beta model works for any underlying credit quality, whereas many familiar models, such as Cox–Ingersoll–Ross [1985], Jamshidian [1989], and Heath–Jarrow–Morton [1992], apply only when the underlying is default-free.[3]

CONCLUSION

Securities markets are not complete and market prices do not follow convenient stochastic processes. Thus, valuation models that rely on complete markets and stochastic price processes often fail in practice. The EPT offers process-free valuation in incomplete markets. In particular, it adopts the maximum-entropy formalism to constructing unique risk-neutral probability densities in incomplete markets.

The gamma model for pricing stock options and the beta model for pricing fixed-income options are products of the EPT. First, the gamma and beta models are valid in incomplete markets and do not restrict prices, interest rates, or volatilities to any specific trajectories. Second, the gamma and beta models are more accurate than alternative option pricing models. Finally, the gamma and beta models are as easy to implement and to use as the Black–Scholes model.

ENDNOTES

I am grateful to Jose Siqueira, the referee, for his comments and suggestions which improved the article.

[1]A securities market is said to be complete if for every future state of the world there is a security (or a portfolio of securities) that pays 1 in that state and 0 in every other state. As a result, every future state can be insured. Otherwise, the market is said to be incomplete.

[2]Black [1976], Brennan [1979], Jamshidian [1989], and other fixed-income models, including very sophisticated ones, generate similar biases. For example, Amin and Morton [1994] reported similar price biases after testing several versions of the Heath–Jarrow–Morton [1992] model.

[3]Fixed-income models usually employ a term structure that has a dual duty, that is, it serves as the underlying "state variable" and it provides risk-free discounting, required by arbitrage-free pricing. Only a default-free term structure can fulfill both duties at once; therefore, the underlying instrument must be default-free. The beta model does not use a term structure and does not restrict the underlying credit quality.

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